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Constraints on effective Lagrangian of D-branes from non-commutative gauge theory

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Abstract

It was argued that there are two different descriptions of the effective Lagrangian of gauge fields on D-branes by non-commutative gauge theory and by ordinary gauge theory in the presence of a constant B field background. In the case of bosonic string theory, however, it was found in the previous works that the two descriptions are incompatible under the field redefinition which relates the non-commutative gauge field to the ordinary one found by Seiberg and Witten. In this paper we resolve this puzzle to observe the necessity of gauge-invariant but B-dependent correction terms involving metric in the field redefinition which have not been considered before. With the problem resolved, we establish a systematic method under the α' expansion to derive the constraints on the effective Lagrangian imposed by the compatibility of the two descriptions where the form of the field redefinition is not assumed.

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1. Introduction

In the light of the recent developments in superstring and M-theory brought by introducing D-branes, it would be impossible to underestimate the importance of understanding the dynamics of collective coordinates of D-branes, such as scalar fields on the worldvolume of the D-branes representing their transverse positions and gauge fields describing internal degrees of freedom. In some situations or limits, the effective Lagrangian describing such collective coordinates is approximated or even supposed to be exactly described by the dimensional reduction of super Yang-Mills theory from ten dimensions to the worldvolume dimensions of the D-branes [1, 2]. For example, the matrix model of M-theory [3] is based on the description of D0-branes in terms of super Yang-Mills theory and the most typical case of the AdS/CFT correspondence [4], namely the correspondence between AdS_5 and the four-dimensional super Yang-Mills theory, is based on that of D3-branes.

Since the perturbative interactions between D-branes and those between D-branes and elementary excitations of strings are completely defined by the open string sigma model with the Dirichlet boundary condition, it is in principle possible to calculate systematic corrections of the effective Lagrangian to the Yang-Mills theory. For example, if we want to obtain the effective Lagrangian of gauge fields on D-branes, we should calculate the S-matrix of the scattering processes of the gauge fields on D-branes in string theory, then construct the effective Lagrangian such that it reproduces the S-matrix correctly. Another way to calculate the effective Lagrangian is to calculate the beta function of the open string sigma model with Dirichlet boundary condition and to look for a Lagrangian whose equation of motion coincides with the condition that the beta function vanishes. The resulting Lagrangian is believed to coincide with the one obtained from the string S-matrix at least for tree-level processes. However, the complexity of the calculation will necessarily increase if we proceed to higher orders in the expansion with respect to α' and the string coupling constant g_s in the S-matrix approach and to higher loops in the beta function approach so that it would be helpful if other complementary approaches to the effective Lagrangian are available.

Recently it is argued that the effective Lagrangian of the gauge fields on D-branes is

described by non-commutative gauge theory [5]-[11] in the presence of a constant background field of the Neveu-Schwarz–Neveu-Schwarz two-form gauge field which is usually referred to as B field. It is also possible to describe it in terms of ordinary gauge theory, however, the B-dependence in the two descriptions is totally different and it turned out that it is possible to constrain the form of the effective Lagrangian by the compatibility of the two descriptions. Actually, it was shown in [12] that the Dirac-Born-Infeld (DBI) Lagrangian [13]-[15] [‡] satisfies the compatibility in the approximation of neglecting derivatives of field strength and its particular form was essential for the compatibility. It is impossible to derive the DBI Lagrangian from the gauge invariance alone so that this shows that the requirement of the compatibility does provide us with information on the dynamics of the gauge fields.

The proof of the equivalence of the two descriptions for the DBI Lagrangian in [12] was beautiful, however it is not clear how we can obtain the constraints for other terms imposed by the compatibility in a systematic way so as to study how powerful and useful this approach will be. This is our basic motivation of the present paper and we will present a method to obtain the constraints systematically in the α' expansion.

Actually a method towards this goal was developed to some extent in [17] where the problem of whether it is possible to include two-derivative corrections[§] to the DBI Lagrangian satisfying the compatibility was discussed and the most general form of the two-derivative corrections up to the quartic order of field strength, F^4, in the α' expansion was derived. However there was a puzzle that the form of the two-derivative terms which is consistent with the compatibility disagreed with the effective Lagrangian derived from bosonic string theory although it was consistent with superstring theory. Does this mean that the equivalence of the two descriptions in the presence of a constant B field fails in bosonic string theory? In the light of the argument in [12], we do not think that it is the case. It is most likely that we had made too strong assumptions so that we only obtained a limited class of Lagrangians which excludes that of bosonic string theory. If it is the case, the methods which are currently available such as the one in [17] do not fulfill our

[‡] For a recent review of the Dirac-Born-Infeld theory see [16] and references therein.

[§] By n-derivative corrections to the DBI Lagrangian, we mean terms with n derivatives acting on field strengths (not on gauge fields).

purpose to derive the constraints correctly. Furthermore the problem may not be limited to the case of bosonic string theory. Without resolving the puzzle, it would be dangerous to develop discussions based on the equivalence of the two descriptions. Therefore we have to reconsider the assumptions which have been made and find out the correct set of the assumptions from which we should derive the constraints using the problem of whether the puzzle in bosonic strings is resolved as a touchstone of the validity of our approach.

It will turn out that the assumption which is not satisfied in bosonic strings is the one on the form of the field redefinition which relates the ordinary gauge field to the non-commutative one. The field redefinition which preserves the gauge equivalence relation found in [12] and further discussed in [18] should be modified in general and suffered from gauge-invariant but B-dependent correction terms involving metric. In particular, our result will show that such terms *must* exist in the case of bosonic string theory. We will argue that the form of the field redefinition should not be assumed as input when constraining the form of the effective Lagrangian and can be rather regarded as a consequence of the compatibility of the two descriptions. This argument is essential in resolving the puzzle in the case of bosonic string theory as we will see. We could jump into the problem of two-derivative correction terms to resolve the puzzle, however, we will first determine the F^4 terms which coincide with those in the DBI Lagrangian correctly without assuming the form of the field redefinition in order to show that the idea presented in this paper is useful to constrain the effective Lagrangian. We then apply it to two-derivative corrections to resolve the puzzle and the generalization to other cases would be straightforward.

The organization of this paper is as follows. In Section 2, we first review the two descriptions of the effective Lagrangian of the gauge fields on D-branes in the presence of a constant B field, namely, the one in terms of ordinary gauge theory and the one by non-commutative gauge theory, to clarify what we assume when deriving the constraints. We then derive the F^4 terms in the DBI Lagrangian without assuming the form of the field redefinition which relates the ordinary gauge field to the non-commutative one in Section 3. We extend our consideration to two-derivative corrections to the DBI Lagrangian in Section 4 where the discrepancy in the case of bosonic string theory is resolved by generalizing the form of the field redefinition. Section 5 is devoted to conclusions and

discussions.

2. Review of the two descriptions in the presence of B

Let us first review the two descriptions of the effective Lagrangian of D-branes in the presence of a constant B field background B_{ij}. In this paper, we concentrate on the effective Lagrangian of a gauge field on a single D-brane in flat space-time, with constant metric g_{ij}, for simplicity.

The worldsheet action describing this system is

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma(g_{ij}\partial_a x^i\partial^a x^j - 2\pi i\alpha' B_{ij}\epsilon^{ab}\partial_a x^i\partial_b x^j) \\ &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma g_{ij}\partial_a x^i\partial^a x^j - \frac{i}{2}\int_{\partial\Sigma} d\tau B_{ij}x^i\partial_\tau x^j, \end{aligned} \tag{2.1}$$

where Σ is the string worldsheet with Euclidean signature and $\partial\Sigma$ is its boundary. A background gauge field couples to the string worldsheet by adding

$$S_{int} = -i\int_{\partial\Sigma} d\tau A_i(x)\partial_\tau x^i \tag{2.2}$$

to the action (2.1). Comparing (2.1) and (2.2), we see that a constant B field can be replaced by the gauge field

$$A_i = -\frac{1}{2}B_{ij}x^j,$$

whose field strength is $F_{ij} = B_{ij}$. Thus we conclude that there exists a definition of a gauge field in the effective Lagrangian such that the effective Lagrangian depends on B and F only in the combination $B+F$ when we turn on a constant B field. This gauge field is an ordinary one, namely, the gauge transformations and its field strength are defined by

$$\delta_\lambda A_i = \partial_i\lambda, \tag{2.3}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i, \tag{2.4}$$

$$\delta_\lambda F_{ij} = 0. \tag{2.5}$$

This is the first description of the effective Lagrangian in terms of ordinary gauge theory.

To derive the second description in terms of non-commutative gauge theory, let us examine the propagator in (2.1). In the presence of a constant B field, the boundary

condition of open strings is modified and is no longer the Neumann one along the D-brane. Thus the propagator in the sigma model is also modified so as to satisfy the new boundary condition. The explicit form of the propagator evaluated at boundary points is [13]-[15]

$$\langle x^i(\tau)x^j(\tau')\rangle = -\alpha'(G^{-1})^{ij}\log(\tau-\tau')^2 + \frac{i}{2}\theta^{ij}\epsilon(\tau-\tau'), \tag{2.6}$$

where the worldsheet is mapped to the upper half plane, τ and τ' are points on the boundary and

$$\begin{aligned} G_{ij} &= g_{ij} - (2\pi\alpha')^2(Bg^{-1}B)_{ij}, && (2.7)\\ \theta^{ij} &= -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}B\frac{1}{g-2\pi\alpha' B}\right)^{ij}. && (2.8) \end{aligned}$$

There are two important modifications here. The first one is the coefficient in front of the log term is no longer the metric $(g^{-1})^{ij}$. The second one is the appearance of the term proportional to the step function $\epsilon(\tau)$ which is 1 or -1 for positive or negative τ.

Now consider the θ-dependence of correlation functions of open string vertex operators which are given by

$$\begin{aligned} &\Big\langle \prod_{n=1}^{k} P_n(\partial x(\tau_n), \partial^2 x(\tau_n), \ldots)e^{ip^n\cdot x(\tau_n)}\Big\rangle_{G,\theta} \\ &= \exp\Big(-\frac{i}{2}\sum_{n>m} p_i^n\theta^{ij}p_j^m\epsilon(\tau_n-\tau_m)\Big) \\ &\qquad\times\Big\langle \prod_{n=1}^{k} P_n(\partial x(\tau_n), \partial^2 x(\tau_n), \ldots)e^{ip^n\cdot x(\tau_n)}\Big\rangle_{G,\theta=0}, \end{aligned} \tag{2.9}$$

where P_n's are polynomials in derivatives of x and x are coordinates along the D-brane. Since the second term in the propagator does not contribute to contractions of derivatives of x, the θ-dependent part can be factorized as the right-hand side of (2.9). The string S-matrix can be obtained from these correlation functions by putting external fields on shell and integrating over the τ's. Therefore, the S-matrix and the effective Lagrangian constructed from it have a structure inherited from this form.

So we can see how the effective Lagrangian is modified when we turn on the constant B field. To distinguish the gauge field in this description from that in the preceding one, let us rename it to $\hat{A}$ and denote the Lagrangian in terms of $\hat{A}$ as $\hat{\mathcal{L}}$. The Lagrangian $\hat{\mathcal{L}}$ is constructed from the one $\mathcal{L}$ in the absence of B as follows.

First, the metric which appears when contracting Lorentz indices is modified to G_{ij} instead of g_{ij} corresponding to the modification in the propagator. Secondly, since the coupling constant can depend on B, let us denote the coupling constant in the presence of B as G_s. Finally, let us go on to the most important modification related to the appearance of the θ-dependent factor

$$\exp\Big(-\frac{i}{2}\sum_{n>m} p_i^n \theta^{ij} p_j^m \epsilon(\tau_n - \tau_m)\Big) \tag{2.10}$$

in (2.9). It corresponds to modifying the ordinary product of functions to the associative but non-commutative $*$ product defined by

$$f(x) * g(x) = \exp\left(\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^j}\right) f(x+\xi)g(x+\zeta)\Big|_{\xi=\zeta=0}, \tag{2.11}$$

in the momentum-space representation. Now the B-dependence of the effective Lagrangian in this description can be obtained through the following replacements: A by $\hat{A}$, g_{ij} by G_{ij}, g_s by G_s and ordinary multiplication by the $*$ product. Corresponding to the modification of the product, the gauge transformations and the definition of field strength are also modified as follows:

$$\hat{\delta}_{\hat{\lambda}}\hat{A}_i = \partial_i\hat{\lambda} + i\hat{\lambda} * \hat{A}_i - i\hat{A}_i * \hat{\lambda}, \tag{2.12}$$

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i - i\hat{A}_i * \hat{A}_j + i\hat{A}_j * \hat{A}_i, \tag{2.13}$$

$$\hat{\delta}_{\hat{\lambda}}\hat{F}_{ij} = i\hat{\lambda} * \hat{F}_{ij} - i\hat{F}_{ij} * \hat{\lambda}. \tag{2.14}$$

We have seen that there are two different effective Lagrangians of the gauge field on the D-brane which reproduce the S-matrix of string theory in the presence of a constant B. What we have learned from the action (2.1) and the interaction (2.2) can be summarized as follows.

1. There exists a definition of a gauge field A_i such that the Lagrangian in terms of it respects the ordinary gauge invariance and it depends on B only in the combination $B + F$.

2. There exists a definition of a gauge field $\hat{A}_i$ such that the Lagrangian in terms of it respects the non-commutative gauge invariance and it depends on B only through

G_{ij}, G_s and θ^{ij} in the non-commutative $*$ product.

$$(2.15)$$

These are our fundamental assumptions and we will consider constraints on the form of the effective Lagrangian imposed by the compatibility of them in what follows.

It is not surprising that there are different descriptions of the effective Lagrangian since the S-matrix is unchanged under field redefinitions in the effective Lagrangian so that the construction of the effective Lagrangian from the S-matrix elements is always subject to an ambiguity originated in the field redefinitions. Thus we do not expect that the two gauge fields A_i and $\hat{A}_i$ coincide: they would be related by a field redefinition. Usually we consider field redefinitions of the form

$$A_i \to A_i + f_i(\partial, F),$$

where $f_i(\partial, F)$ denotes an arbitrary gauge-invariant expression made of F_{ij}, $\partial_k F_{ij}$, $\partial_k \partial_l F_{ij}$, and so on. The field redefinitions of this kind preserve the ordinary gauge invariance. However they will not work in this case because the gauge transformation of $\hat{A}_i$ is different from that of A_i. The field redefinition which relates $\hat{A}_i$ to A_i must preserve the gauge equivalence relation, namely it satisfies

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}} \hat{A}(A) = \hat{A}(A + \delta_\lambda A), \tag{2.16}$$

with infinitesimal λ and $\hat{\lambda}$. Whether there exists a field redefinition which satisfies (2.16) is a nontrivial question, however, a perturbative solution with respect to θ was found by Seiberg and Witten [12]. Its explicit form for the rank-one case is given by¶

$$\hat{A}_i = A_i - \frac{1}{2}\theta^{kl} A_k (\partial_l A_i + F_{li}) + O(\theta^2), \tag{2.17}$$

$$\hat{\lambda} = \lambda + \frac{1}{2}\theta^{kl} \partial_k \lambda A_l + O(\theta^2). \tag{2.18}$$

However we should emphasize here that *we do not assume the explicit form of the field redefinition which relates $\hat{A}_i$ to A_i when we derive constraints on the form of the effective Lagrangian in the present paper.* What we assume is the two assumptions (2.15) alone.

¶ Solutions to the gauge equivalence relation were further discussed in [18].

This is an important difference from the previous works such as [12] or [17]. The form of the field redefinition is rather regarded as a consequence of the compatibility of the two descriptions in terms of ordinary and non-commutative gauge theories as we will see in the next section.

Before proceeding, we should make a comment on the relation between our assumptions (2.15) and regularization schemes in the sigma model. We mentioned the ambiguity related to field redefinitions in constructing effective Lagrangian from S-matrix elements. In the case of string theory, we can also understand the origin of the ambiguity in the point of view of the sigma model to be coming from degrees of freedom to choose different regularization schemes as was discussed in [12]. We arrived at the assumptions (2.15) from the properties of (2.1) and (2.2) at classical level. However it is necessary to regularize the theory to define composite operators such as (2.2) at quantum level. The description in terms of the ordinary gauge field A_i will be derived from a Pauli-Villars type regularization while the description in terms of the non-commutative gauge field $\hat{A}_i$ will be derived from a point-splitting type regularization. However if we take the simple point-splitting regularization discussed in [12] in which we cut out the region $|\tau - \tau'| < \delta$ and take the limit $\delta \to 0$, the non-commutative gauge transformation suffers from α' corrections before taking the zero slope limit. Therefore it is not clear whether there is an appropriate regularization corresponding to the non-commutative gauge field $\hat{A}_i$ in the second assumption of (2.15) where no zero slope limit is taken. In this sense, we regard (2.15) as assumptions although we can argue that they are plausible in the following way. If the effective action before turning on B is invariant under the ordinary gauge transformation and the B-dependence can be made only through G_{ij}, G_s and θ^{ij}, the action after turning on B is automatically invariant under the non-commutative gauge transformation (2.12) at least for the case where the rank of the gauge group is greater than one. The case with the rank-one gauge theory may be slightly subtle but it would be naturally expected that it holds in this case as well. At any rate, our basic standpoint is that the effective Lagrangian we discuss in this paper is constructed so as to reproduce the S-matrix elements correctly and it is not necessary to consider its relation to the background field in the sigma model in what follows.

3. Determination of F^4 terms revisited

3.1 Determination without assuming the form of the field redefinition

Let us now proceed to see how the F^4 terms in the DBI Lagrangian are determined by the assumptions (2.15) although the form of the field redefinition is not assumed. Since we study the effective Lagrangian in the α' expansion, we present the following formulas for convenience which will be used repeatedly:

$$
\begin{aligned}
(G^{-1})^{ij} &= (g^{-1})^{ij} + (2\pi\alpha')^2(g^{-1}Bg^{-1}Bg^{-1})^{ij} + O(\alpha'^4), && (3.1)\\
\theta^{ij} &= -(2\pi\alpha')^2(g^{-1}Bg^{-1})^{ij} + O(\alpha'^4), && (3.2)\\
f * g &= fg - \frac{i}{2}(2\pi\alpha')^2(g^{-1}Bg^{-1})^{kl}\partial_k f \partial_l g + O(\alpha'^4). && (3.3)
\end{aligned}
$$

The lowest order term of the effective Lagrangian of a gauge field on a D-brane in the α' expansion is the F^2 term:

$$
\begin{aligned}
\mathcal{L}(F) &= \frac{\sqrt{\det g}}{g_s}\left[(g^{-1})^{ij}F_{jk}(g^{-1})^{kl}F_{li} + O(\alpha')\right] \\
&= \frac{\sqrt{\det g}}{g_s}\left[F_{ij}F_{ji} + O(\alpha')\right] \\
&\equiv \frac{\sqrt{\det g}}{g_s}\left[\mathrm{Tr}F^2 + O(\alpha')\right]. \qquad (3.4)
\end{aligned}
$$

Here we omitted a possible overall factor including an appropriate power of α'. Since the discussions presented in this paper do not depend on the dimension of space-time on which the gauge theory is defined, namely, the dimension of worldvolume of the D-brane, if we want to supply the overall factor, we only need to multiply an appropriate power of α' to the Lagrangian to make the action dimensionless and a numerical constant which depends on the convention. In the second line of (3.4), we made g^{-1} implicit as

$$A_iB_i \equiv (g^{-1})^{ij}A_iB_j. \qquad (3.5)$$

Since Lorentz indices in most of the expressions in what follows are contracted with respect to the metric g_{ij}, we will adopt this convention together with

$$\partial^2 \equiv (g^{-1})^{ij}\partial_i\partial_j, \qquad (3.6)$$

to simplify the expressions unless the other metric G_{ij} is explicitly used. And Tr denotes the trace over Lorentz indices as can be seen from the third line of (3.4).

Now the assumptions (2.15) imply that we can describe the system in two different ways when we turn on B as follows:

$$\mathcal{L}(B+F) = \frac{\sqrt{\det g}}{g_s}\left[\mathrm{Tr}(B+F)^2 + O(\alpha')\right], \tag{3.7}$$

$$\hat{\mathcal{L}}(\hat{F}) = \frac{\sqrt{\det G}}{G_s}\left[(G^{-1})^{ij}\hat{F}_{jk} * (G^{-1})^{kl}\hat{F}_{li} + O(\alpha')\right]. \tag{3.8}$$

In the case of higher-rank gauge theory, it follows from the comparison between (3.7) and (3.8) when B vanishes that

$$G_s = g_s + O(\alpha'), \tag{3.9}$$

$$\hat{A}_i = A_i + O(\alpha'). \tag{3.10}$$

In the rank-one case, on the other hand, we can only determine the normalizations of G_s and $\hat{A}_i$ as

$$G_s = t g_s + O(\alpha'), \tag{3.11}$$

$$\hat{A}_i = \sqrt{t} A_i + O(\alpha'), \tag{3.12}$$

from the consideration at the lowest order in α' alone since there is no interaction in the F^2 term. The normalizations of A_i and $\hat{A}_i$ and hence that of G_s are already determined by (2.15) since if we rescale A_i or $\hat{A}_i$ then the B-dependence does not take the combination $B+F$ for the description in terms of A_i and the field strength $\hat{F}_{ij}$ does not take the form (2.13) anymore as for the description using $\hat{A}_i$. Therefore we can in principle determine the constant t from the assumptions (2.15). However the calculation for the determination of t is slightly messy so that we will defer it to Appendix A and proceed assuming $t = 1$ in this section for the sake of brevity which will be justified in Appendix A.

Let us first check that $\mathcal{L}(B+F)$ and $\hat{\mathcal{L}}(\hat{F})$ coincide at the lowest order in α', which is necessary to be consistent with (2.15). In general, the Lagrangian $\hat{\mathcal{L}}$ on the non-commutative side reduces to the one $\mathcal{L}$ on the commutative side at the lowest order in α'. In this case,

$$(G^{-1})^{ij}\hat{F}_{jk} * (G^{-1})^{kl}\hat{F}_{li} = (\partial_i\hat{A}_j - \partial_j\hat{A}_i)(\partial_j\hat{A}_i - \partial_i\hat{A}_j) + O(\alpha'^2). \tag{3.13}$$

What is less trivial is the question whether $\mathrm{Tr}(B+F)^2$ reduces to $\mathrm{Tr}F^2$ up to total derivative, namely, whether $\mathrm{Tr}F^2$ satisfies the *initial term condition* defined by

$$f(B+F) = f(F) + \text{total derivative}, \tag{3.14}$$

in [17], which is the condition for a term to be qualified as an initial term of a consistent Lagrangian in the α' expansion. It is verified that $\mathrm{Tr}F^2$ satisfies this condition as follows:

$$\begin{aligned} & \mathrm{Tr}(B+F)^2 \\ = \; & \mathrm{Tr}F^2 + 2\mathrm{Tr}BF + \mathrm{Tr}B^2 \\ = \; & \mathrm{Tr}F^2 + \text{total derivative} + \text{const.} \end{aligned} \tag{3.15}$$

The F^4 terms in the DBI Lagrangian are determined by the consideration at the next order terms in the α' expansion of (3.8), which are given by

$$\begin{aligned} & \frac{\sqrt{\det G}}{G_s}(G^{-1})^{ij}\hat{F}_{jk} * (G^{-1})^{kl}\hat{F}_{li} \\ = \; & \frac{\sqrt{\det g}}{G_s}\Big[(\partial_i\hat{A}_j - \partial_j\hat{A}_i)(\partial_j\hat{A}_i - \partial_i\hat{A}_j) - 4(2\pi\alpha')^2 B_{kl}\partial_k\hat{A}_i\partial_l\hat{A}_j\partial_j\hat{A}_i \\ & +2(2\pi\alpha')^2(B^2)_{ij}(\partial_j\hat{A}_k - \partial_k\hat{A}_j)(\partial_k\hat{A}_i - \partial_i\hat{A}_k) \\ & -\frac{1}{2}(2\pi\alpha')^2\mathrm{Tr}B^2(\partial_i\hat{A}_j - \partial_j\hat{A}_i)(\partial_j\hat{A}_i - \partial_i\hat{A}_j) + O(\alpha'^4)\Big]. \end{aligned} \tag{3.16}$$

What is important here is the existence of the second term on the right-hand side of (3.16). It gives a non-vanishing contribution to the three-point scattering amplitude of the gauge fields. More precisely, if we represent the asymptotic fields in N-point scattering as

$$A_i^{\text{asym } a}(x) = \zeta_i^a e^{ik^a\cdot x}, \qquad a = 1, 2, \ldots, N, \tag{3.17}$$

$$(k^a)^2 = 0, \quad \zeta^a \cdot k^a = 0, \quad \sum_{a=1}^{N} k_i^a = 0, \tag{3.18}$$

the second term on the right-hand side of (3.16) gives a contribution to the three-point amplitude of order $O(B, \zeta^3, k^3)$. It can be easily shown that no other term can produce the contribution of this form on the non-commutative side. Therefore this contribution cannot be canceled and must be reproduced from the Lagrangian $\mathcal{L}(B+F)$ on the commutative side.

There are two terms on the commutative side which can produce the $O(B, \zeta^3, k^3)$ contribution to the three-point amplitude. They are $\mathrm{Tr}(B+F)^4$ and $[\mathrm{Tr}(B+F)^2]^2$:

$$\begin{aligned}\mathrm{Tr}(B+F)^4 &= \mathrm{Tr}F^4 + 4\mathrm{Tr}BF^3 + O(B^2), &(3.19)\\ \left[\mathrm{Tr}(B+F)^2\right]^2 &= (\mathrm{Tr}F^2)^2 + 4\mathrm{Tr}BF\mathrm{Tr}F^2 + O(B^2). &(3.20)\end{aligned}$$

There are several terms in $\mathrm{Tr}BF^3$ and $\mathrm{Tr}BF\mathrm{Tr}F^2$ when we expand them as $F_{ij} = \partial_i A_j - \partial_j A_i$, but some of them which contain $\partial^2 A_i$ or $\partial_i A_i$ do not contribute to the S-matrix of the three-point scattering because of the on-shell conditions $k^2 = 0$ and $\zeta \cdot k = 0$. Moreover, it will be useful to observe that terms of the form $f\partial_i g \partial_i h$ in general do not contribute to the S-matrix of three-point scattering where f, g and h are massless fields or their derivatives. This follows from the fact $k^1 \cdot k^2 = k^2 \cdot k^3 = k^3 \cdot k^1 = 0$ which can be easily seen as

$$0 = (k^3)^2 = (k^1 + k^2)^2 = 2k^1 \cdot k^2, \tag{3.21}$$

where we used $k^1 + k^2 + k^3 = 0$ and $(k^a)^2 = 0$. Another way to see this is to rewrite $f\partial_i g \partial_i h$ as follows:

$$f\partial_i g \partial_i h = \frac{1}{2}(\partial^2 f gh - f\partial^2 gh - fg\partial^2 h) + \frac{1}{2}\partial^2(fgh) - \partial_i(\partial_i fgh). \tag{3.22}$$

Having been equipped with this formula, we can extract the part which contributes to the S-matrix from $\mathrm{Tr}BF^3$ and $\mathrm{Tr}BF\mathrm{Tr}F^2$ as follows:

$$\begin{aligned}\mathrm{Tr}BF^3 &= B_{ij}F_{jk}F_{kl}F_{li}\\ &= 2B_{ij}\partial_j A_k \partial_k A_l \partial_l A_i - 2B_{ij}\partial_j A_k \partial_k A_l \partial_i A_l\\ &\quad + \text{terms with } \partial^2 A + \text{total derivative}, &(3.23)\\ \mathrm{Tr}BF\mathrm{Tr}F^2 &= B_{ij}F_{ji}F_{kl}F_{lk}\\ &= 4B_{ij}\partial_j A_i \partial_k A_l \partial_l A_k\\ &= -8B_{ij}A_i \partial_k A_l \partial_j \partial_l A_k + \text{total derivative}\\ &= 8B_{ij}\partial_l A_i \partial_k A_l \partial_j A_k + \text{a term with } \partial_l A_l + \text{total derivative}. &(3.24)\end{aligned}$$

To summarize, we have found that all the terms which contribute to the S-matrix of order $O(B, \zeta^3, k^3)$. On the non-commutative side, there was only one source,

$$\mathrm{Tr}(G^{-1}\hat{F} * G^{-1}\hat{F}) \to -4(2\pi\alpha')^2 B_{kl}\partial_k \hat{A}_i \partial_l \hat{A}_j \partial_j \hat{A}_i,$$

while there were two on the commutative side:

$$
\begin{aligned}
\mathrm{Tr}BF^3 &\rightarrow 2B_{ij}\partial_j A_k \partial_k A_l \partial_l A_i - 2B_{ij}\partial_i A_l \partial_j A_k \partial_k A_l, \\
\mathrm{Tr}BF\mathrm{Tr}F^2 &\rightarrow 8B_{ij}\partial_j A_k \partial_k A_l \partial_l A_i.
\end{aligned}
$$

It is not difficult to show that the contributions to the S-matrix from $B_{ij}\partial_j A_k \partial_k A_l \partial_l A_i$ and $B_{ij}\partial_i A_l \partial_j A_k \partial_k A_l$ are non-vanishing and linearly independent. Thus the conclusion derived from (2.15) is that to reproduce the contribution to the S-matrix from the Lagrangian $\hat{\mathcal{L}}(\hat{F})$, the following terms must exist in the Lagrangian $\mathcal{L}(B+F)$:

$$
2(2\pi\alpha')^2 \mathrm{Tr}BF^3 - \frac{1}{2}(2\pi\alpha')^2 \mathrm{Tr}BF\mathrm{Tr}F^2. \tag{3.25}
$$

We can uniquely construct the Lagrangian $\mathcal{L}(F)$ such that $\mathcal{L}(B+F)$ generates the terms (3.25), which is given by

$$
\begin{aligned}
\mathcal{L}(F) = \frac{\sqrt{\det g}}{g_s}\Bigg[& \mathrm{Tr}F^2 + (2\pi\alpha')^2 \left[\frac{1}{2}\mathrm{Tr}F^4 - \frac{1}{8}(\mathrm{Tr}F^2)^2\right] \\
& + O(\alpha'^4) + \text{derivative corrections}\Bigg].
\end{aligned} \tag{3.26}
$$

This coincides with the α' expansion of the DBI Lagrangian for a single Dp-brane,

$$
\mathcal{L}_{DBI}(F) = \frac{1}{g_s(2\pi)^p(\alpha')^{(p+1)/2}}\sqrt{\det(g+2\pi\alpha' F)}, \tag{3.27}
$$

up to an overall factor and an additive constant. Thus we have succeeded in determining the F^4 terms in the DBI Lagrangian from the assumptions (2.15) without referring to the explicit form of the field redefinition which relates $\hat{A}_i$ to A_i. We will derive its form in the next subsection.

3.2 Field redefinition

We have seen that the two effective Lagrangians,

$$
\begin{aligned}
\mathcal{L}(B+F) = \frac{\sqrt{\det g}}{g_s}\Bigg[& \mathrm{Tr}(B+F)^2 + (2\pi\alpha')^2 \left[\frac{1}{2}\mathrm{Tr}(B+F)^4 - \frac{1}{8}[\mathrm{Tr}(B+F)^2]^2\right] \\
& + O(\alpha'^4) + \text{derivative corrections}\Bigg],
\end{aligned} \tag{3.28}
$$

and

$$\hat{\mathcal{L}}(\hat{F}) = \frac{\sqrt{\det G}}{G_s}\Bigg[\mathrm{Tr}(G^{-1}\hat{F} * G^{-1}\hat{F}) + (2\pi\alpha')^2\Big[\frac{1}{2}\mathrm{Tr}(G^{-1}\hat{F})^4_{\text{arbitrary}} -\frac{1}{8}(\mathrm{Tr}(G^{-1}\hat{F})^2)^2_{\text{arbitrary}}\Big] + \ O(\alpha'^4) + \text{derivative corrections}\ \Bigg], \tag{3.29}$$

produce the same contribution to the S-matrix of order $O(B,\zeta^3,k^3)$. Here we added the $\hat{F}^4$ terms to $\hat{\mathcal{L}}(\hat{F})$ which were required by the existence of the corresponding F^4 terms in $\mathcal{L}(B+F)$ and the subscripts "arbitrary" there mean that the ordering of the four field strengths in each term is arbitrary. Since the $*$ product is non-commutative, we have to specify the ordering of field strengths as in the case of the ordinary Yang-Mills theory. However, there is no principle in determining the ordering for the rank-one case and we leave it arbitrary for now. The fact that two effective Lagrangians produce the same contribution to the S-matrix does not mean that the two must coincide at off-shell level but implies that the fields in the two effective Lagrangians can be related by a field redefinition. Let us see this explicitly for the case in hand. By expanding the $O(\alpha'^2)$ terms in $\mathcal{L}(B+F)$, we have

$$\begin{aligned}
& \frac{1}{2}(2\pi\alpha')^2\mathrm{Tr}(B+F)^4 - \frac{1}{8}(2\pi\alpha')^2[\mathrm{Tr}(B+F)^2]^2 \\
= \ & \frac{1}{2}(2\pi\alpha')^2\mathrm{Tr}F^4 - \frac{1}{8}(2\pi\alpha')^2(\mathrm{Tr}F^2)^2 \\
& +2(2\pi\alpha')^2\mathrm{Tr}BF^3 - \frac{1}{2}(2\pi\alpha')^2\mathrm{Tr}BF\mathrm{Tr}F^2 \\
& +2(2\pi\alpha')^2\mathrm{Tr}B^2F^2 - \frac{1}{4}(2\pi\alpha')^2\mathrm{Tr}B^2\mathrm{Tr}F^2 + \text{total derivative} + \text{const.},
\end{aligned} \tag{3.30}$$

where we used the fact that

$$(2\pi\alpha')^2\left[\mathrm{Tr}(BF)^2 - \frac{1}{2}(\mathrm{Tr}BF)^2\right] = \text{total derivative}. \tag{3.31}$$

Obviously the $O(B)$ and $O(B^2)$ parts of (3.30) do not coincide with those of (3.16) if we assume $\hat{A}_i = A_i$. Let us first consider the difference in the $O(B)$ part:

$$\begin{aligned}
\Delta\mathcal{L} \ \equiv \ & 2(2\pi\alpha')^2\mathrm{Tr}BF^3 - \frac{1}{2}(2\pi\alpha')^2\mathrm{Tr}BF\mathrm{Tr}F^2 - \left(-4(2\pi\alpha')^2 B_{kl}\partial_k A_i \partial_l A_j \partial_j A_i\right) \\
= \ & 2(2\pi\alpha')^2 B_{kl}F_{lj}F_{ji}F_{ik} + 2(2\pi\alpha')^2 B_{kl}A_k\partial_l F_{ij}F_{ji} + 2(2\pi\alpha')^2 B_{kl}\partial_k A_i\partial_l A_j F_{ji} \\
& + \text{total derivative} \\
= \ & 2(2\pi\alpha')^2 B_{kl}F_{ji}(F_{lj}F_{ik} + A_k\partial_l F_{ij} + \partial_k A_i \partial_l A_j) + \text{total derivative}.
\end{aligned} \tag{3.32}$$

This must be reduced to the field redefinition which relates $\hat{A}_i$ to A_i. We can make it manifest by noting the fact that

$$\begin{aligned}
& B_{kl}F_{ji}\partial_i[A_k(\partial_l A_j + F_{lj})] \\
= \;& B_{kl}F_{ji}[\partial_i A_k(\partial_l A_j + F_{lj}) + A_k(\partial_l\partial_i A_j + \partial_i F_{lj})] \\
= \;& B_{kl}F_{ji}\left[(F_{ik} + \partial_k A_i)(F_{lj} + \partial_l A_j) + A_k\left(\frac{1}{2}\partial_l F_{ij} + \partial_i F_{lj}\right)\right] \\
= \;& B_{kl}F_{ji}(F_{lj}F_{ik} + A_k\partial_l F_{ij} + \partial_k A_i\partial_l A_j),
\end{aligned} \tag{3.33}$$

where we used the facts that

$$F_{ji}\partial_i F_{lj} = \frac{1}{2}F_{ji}\partial_l F_{ij}, \tag{3.34}$$

and that

$$B_{kl}F_{ji}(F_{ik}\partial_l A_j + \partial_k A_i F_{lj}) = 0. \tag{3.35}$$

Then the difference $\Delta\mathcal{L}$ can be rewritten using (3.33) as

$$\begin{aligned}
\Delta\mathcal{L} &= 2(2\pi\alpha')^2 B_{kl}F_{ji}\partial_i[A_k(\partial_l A_j + F_{lj})] + \text{total derivative} \\
&= 2(2\pi\alpha')^2 B_{kl}\partial_i F_{ij}A_k(\partial_l A_j + F_{lj}) + \text{total derivative}.
\end{aligned} \tag{3.36}$$

The fact that $\Delta\mathcal{L}$ does not contribute to the S-matrix and can be reduced to the field redefinition of $\hat{A}_i$ is now manifest in this form since $\Delta\mathcal{L}$ is proportional to $\partial_i F_{ij}$ and hence vanishes using the equation of motion. If we write

$$\hat{A}_i = A_i + (2\pi\alpha')^2\Delta A_i + O(\alpha'^4), \tag{3.37}$$

it obeys that

$$(\partial_i\hat{A}_j - \partial_j\hat{A}_i)(\partial_j\hat{A}_i - \partial_i\hat{A}_j) = F_{ij}F_{ji} + 4(2\pi\alpha')^2\partial_i F_{ij}\Delta A_j + O(\alpha'^4). \tag{3.38}$$

Thus the appropriate field redefinition is determined by solving the equation

$$4(2\pi\alpha')^2\partial_i F_{ij}\Delta A_j = \Delta\mathcal{L}. \tag{3.39}$$

The solution is given by

$$\Delta A_i = \frac{1}{2}B_{kl}A_k(\partial_l A_i + F_{li}), \tag{3.40}$$

up to gauge transformations, and the relation between $\hat{A}_i$ and A_i is

$$\hat{A}_i = A_i + \frac{1}{2}(2\pi\alpha')^2 B_{kl} A_k (\partial_l A_i + F_{li}) + O(\alpha'^4). \tag{3.41}$$

This precisely coincides with the field redefinition (2.17) found by Seiberg and Witten [12] if we express θ in terms of B. This was expected since we assumed in (2.15) the ordinary gauge invariance in the description in terms of A_i and the non-commutative gauge invariance in the description using $\hat{A}_i$ so that the gauge equivalence relation (2.16) must be satisfied. Our result is therefore consistent with the previous works. However it is important to note that this form of the field redefinition should be regarded as a consequence of the assumptions (2.15) in our approach. We did not have to know the form of the field redefinition in the determination of the F^4 terms and the form of the field redefinition was determined from the difference between the two effective Lagrangians at off-shell level.

The $O(B^2)$ part of the difference between (3.16) and (3.30),

$$\frac{1}{4}(2\pi\alpha')^2 \mathrm{Tr} B^2 \mathrm{Tr} F^2, \tag{3.42}$$

is proportional to the F^2 term so that it can be absorbed into the definition of G_s as follows:

$$G_s = g_s \left[1 - \frac{1}{4}(2\pi\alpha')^2 \mathrm{Tr} B^2 + O(\alpha'^4) \right]. \tag{3.43}$$

Here it is also possible to take care of the difference (3.42) by a field redefinition of $\hat{A}_i$ just as in the case of the difference in the $O(B)$ part and we cannot determine how we should treat (3.42) from the consideration at the order α'^2. However since the normalizations of A_i and $\hat{A}_i$ are already determined by (2.15) as we mentioned below (3.12), the ambiguity must be fixed by the consideration at higher orders. We will determine the $O(\alpha'^2)$ part of G_s in Appendix B from the consideration at order α'^4, which justifies (3.43).

We have demonstrated how to constrain the effective Lagrangian of gauge fields on D-branes from the assumptions (2.15) for the F^4 terms in the DBI Lagrangian. We should now proceed to the reconsideration of the constraints on the two-derivative corrections to the DBI Lagrangian where the discrepancy was found in the case of bosonic string theory [17].

4. Constraints on two-derivative corrections

4.1 $O(\alpha')$ terms

The two-derivative corrections to the DBI Lagrangian can first appear at order α' compared with the F^2 term. Let us first survey possible terms at this order in both ordinary and non-commutative gauge theories.

In ordinary gauge theory, Lagrangians are made of field strength and its derivatives. At order α', terms of the forms $\partial F \partial F$, $F\partial^2 F$ and F^3 are possible. However since the $F\partial^2 F$ terms can be transformed to the $\partial F \partial F$ terms using the integration by parts and F^3 terms vanish for the rank-one case, it is sufficient to consider the $\partial F \partial F$ terms. There are three different ways to contract Lorentz indices:

$$T_1 \equiv \partial_i F_{ik} \partial_j F_{jk}, \qquad T_2 \equiv \partial_j F_{ik} \partial_i F_{jk}, \qquad T_3 \equiv \partial_k F_{ij} \partial_k F_{ji}. \tag{4.1}$$

Using the Bianchi identity, the term T_3 reduces to T_2,

$$T_3 = -2T_2, \tag{4.2}$$

and the two remaining terms T_1 and T_2 coincide up to total derivative:

$$T_1 = -F_{ik}\partial_i \partial_j F_{jk} + \text{total derivative}, \tag{4.3}$$

$$T_2 = -F_{ik}\partial_j \partial_i F_{jk} + \text{total derivative}. \tag{4.4}$$

Thus any term at order α' can be transformed to T_1.

The story is slightly different in non-commutative gauge theory. The building blocks of Lagrangians in non-commutative gauge theory are field strength $\hat{F}$ and its covariant derivatives defined by

$$\hat{D}_i \hat{F}_{jk} = \partial_i \hat{F}_{jk} - i\hat{A}_i * \hat{F}_{jk} + i\hat{F}_{jk} * \hat{A}_i. \tag{4.5}$$

At order α', terms of the forms $\hat{D}\hat{F}\hat{D}\hat{F}$, $\hat{F}\hat{D}^2\hat{F}$ and $\hat{F}^3$ are possible. The $\hat{F}\hat{D}^2\hat{F}$ terms can be transformed to the $\hat{D}\hat{F}\hat{D}\hat{F}$ terms using the integration by parts as in the case of ordinary gauge theory, but $\hat{F}^3$ terms no longer vanish even for the rank-one case. Thus

there are four terms at order α':[‖]

$$\hat{T}_1 \equiv \hat{D}_i\hat{F}_{ik} * \hat{D}_j\hat{F}_{jk}, \qquad \hat{T}_2 \equiv \hat{D}_j\hat{F}_{ik} * \hat{D}_i\hat{F}_{jk}, \qquad \hat{T}_3 \equiv \hat{D}_k\hat{F}_{ij} * \hat{D}_k\hat{F}_{ji},$$
$$\hat{T}_4 \equiv i\hat{F}_{ij} * \hat{F}_{jk} * \hat{F}_{ki}, \tag{4.6}$$

where we multiplied the $\hat{F}^3$ term by i to make it Hermitian. Using the Bianchi identity, the term $\hat{T}_3$ reduces to $\hat{T}_2$ as before,

$$\hat{T}_3 = -2\hat{T}_2, \tag{4.7}$$

but the terms $\hat{T}_1$ and $\hat{T}_2$ do not coincide up to total derivative since

$$\hat{T}_1 = -\hat{F}_{ik} * \hat{D}_i\hat{D}_j\hat{F}_{jk} + \text{total derivative}, \tag{4.8}$$
$$\hat{T}_2 = -\hat{F}_{ik} * \hat{D}_j\hat{D}_i\hat{F}_{jk} + \text{total derivative}, \tag{4.9}$$

where $\hat{D}_i$ and $\hat{D}_j$ no longer commute. The remaining three terms $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_4$ are not independent which can be seen as follows:

$$\begin{aligned}\hat{T}_1 - \hat{T}_2 &= -\hat{F}_{ik} * [\hat{D}_i, \hat{D}_j]\hat{F}_{jk} + \text{total derivative} \\ &= -\hat{F}_{ik} * (-i\hat{F}_{ij} * \hat{F}_{jk} + i\hat{F}_{jk} * \hat{F}_{ij}) + \text{total derivative} \\ &= -2\hat{T}_4 + \text{total derivative}.\end{aligned} \tag{4.10}$$

We will choose $\{\hat{T}_1, \hat{T}_4\}$ as a basis of $O(\alpha')$ terms in non-commutative gauge theory.

The origin of the extra term $\hat{T}_4$ can be interpreted as an ambiguity in constructing non-commutative gauge theory from ordinary gauge theory for the rank-one case. This can be seen manifestly if we rewrite $\hat{T}_4$ as

$$\hat{T}_4 = \frac{i}{2}\hat{F}_{ij} * (\hat{F}_{jk} * \hat{F}_{ki} - \hat{F}_{ki} * \hat{F}_{jk}) = \frac{1}{2}\hat{F}_{ij} * [\hat{D}_k, \hat{D}_i]\hat{F}_{jk}, \tag{4.11}$$

which precisely corresponds to the ambiguity of the ordering of covariant derivatives when we construct a non-commutative counterpart of the term $F_{ij}\partial_k\partial_i F_{jk}$. This is characteristic of the rank-one theory and there is no such ambiguity in higher-rank cases where the ordering of field strengths or covariant derivatives is already determined in ordinary Yang-Mills theory.

[‖] Lorentz indices on the non-commutative side should be regarded as being contracted using G_{ij} although we will not write it explicitly in this subsection contrary to the conventions (3.5) and (3.6).

We have found bases of $O(\alpha')$ terms for both ordinary and non-commutative gauge theories. We will next consider the properties of the bases with respect to their behavior under field redefinitions and to the relation to our assumptions (2.15).

For ordinary gauge theory our basis consists of T_1 alone. It is possible to absorb T_1 into the F^2 term by a field redefinition which is given by

$$\begin{aligned} \tilde{A}_i &= A_i + a(2\pi\alpha')\partial_j F_{ji} + O(\alpha'^2), && (4.12) \\ \tilde{F}_{ij}\tilde{F}_{ji} &= F_{ij}F_{ji} + 4a(2\pi\alpha')\partial_i F_{ij}\partial_k F_{kj} + \text{total derivative} + O(\alpha'^2). && (4.13) \end{aligned}$$

It is important to notice that this field redefinition has the following property:

$$(B + \tilde{F})_{ij} = (B + F)_{ij} + a(2\pi\alpha')\partial^2(B + F)_{ij} + O(\alpha'^2). \tag{4.14}$$

This implies that if the effective Lagrangian in terms of $\tilde{A}_i$ depends on B only in the form of $B + F$, the Lagrangian in terms of A_i also depends on B only in the combination $B + F$, namely, both A_i and $\tilde{A}_i$ satisfy the first assumption of (2.15). As can be seen from this example, the first assumption of (2.15) does not determine the definition of the gauge field uniquely. For instance, field redefinitions of the form

$$\tilde{A}_i = A_i + f_i(\partial F, \partial^2 F, \ldots), \tag{4.15}$$

where field strengths in f_i are accompanied by at least one derivative, do not spoil the first assumption of (2.15). Since the term T_1 satisfies the initial term condition (3.14) because of the fact that $\partial_i(B + F)_{jk} = \partial_i F_{jk}$ for a constant B, we can proceed allowing a finite T_1 term to be present in the Lagrangian without restricting the definition of the gauge field further. However we will take an alternative approach that we choose a definition of the gauge field in terms of which the T_1 term vanishes in the Lagrangian among the ones which satisfy the first assumption of (2.15) for convenience.

For non-commutative gauge theory our basis consists of $\hat{T}_1$ and $\hat{T}_4$. As in the case of ordinary gauge theory, the term $\hat{T}_1$ can be absorbed into the $\hat{F}^2$ term by a field redefinition given by

$$\begin{aligned} \tilde{\hat{A}}_i &= \hat{A}_i + a(2\pi\alpha')\hat{D}_j \hat{F}_{ji} + O(\alpha'^2), && (4.16) \\ \tilde{\hat{F}}_{ij} * \tilde{\hat{F}}_{ji} &= \hat{F}_{ij} * \hat{F}_{ji} + 4a(2\pi\alpha')\hat{D}_i \hat{F}_{ij} * \hat{D}_k \hat{F}_{kj} + \text{total derivative} + O(\alpha'^2). && (4.17) \end{aligned}$$

This field redefinition preserves the second assumption of (2.15) so that we can select a definition of the non-commutative gauge field satisfying (2.15) such that the term $\hat{T}_1$ vanishes in the Lagrangian. With this convention and the one for the ordinary gauge field we mentioned in the last paragraph, there is no $O(\alpha')$ term in $\mathcal{L}(B+F)$ and only the $\hat{T}_4$ term exists in $\hat{\mathcal{L}}(\hat{F})$ at order α', which implies that

$$\hat{A}_i = A_i + O(\alpha'^2), \tag{4.18}$$

namely, no $O(\alpha')$ part in the field redefinition.

On the other hand the term $\hat{T}_4$ cannot be redefined away and it gives a non-vanishing contribution to the S-matrix at $O(B)$ as we will see shortly. It would be rather trivial that the existence of $\hat{T}_4$ in the effective Lagrangian is consistent with our assumptions (2.15) for the rank-one case since it vanishes in the commutative limit. Incidentally, the term $\hat{T}_4$ is consistent for higher-rank cases as well since its commutative counterpart $i\,\mathrm{trTr}F^3$, where tr denotes the trace over color indices, satisfies the initial term condition (3.14), which can be shown as follows:

$$\begin{aligned} i\,\mathrm{trTr}(B+F)^3 &= \frac{i}{2}\mathrm{tr}(B+F)_{ij}[(B+F)_{jk},(B+F)_{ki}] \\ &= \frac{i}{2}\mathrm{tr}F_{ij}[F_{jk},F_{ki}] + \frac{i}{2}B_{ij}\mathrm{tr}[F_{jk},F_{ki}] \\ &= i\,\mathrm{trTr}F^3. \end{aligned} \tag{4.19}$$

4.2 Constraints on two-derivative corrections

In Section 3.1, we have shown that the $\hat{F}^2$ term produces a non-vanishing contribution to the S-matrix of order $O(B,\zeta^3,k^3)$ and that the F^4 terms are determined by the requirement that the Lagrangian $\mathcal{L}(B+F)$ should reproduce the contribution. Having understood that the term $\hat{T}_4$ is possible at order α', let us develop a similar discussion for two-derivative corrections.

The term $\hat{T}_4$ is evaluated in the α' expansion as follows:

$$\hat{T}_4 = \frac{1}{2}(2\pi\alpha')^2 B_{nm}\hat{F}_{ij}\partial_n\hat{F}_{jk}\partial_m\hat{F}_{ki} + O(\alpha'^4). \tag{4.20}$$

We can extract the part which gives a non-vanishing contribution to the three-point amplitude using the formula (3.22). The result is

$$\hat{T}_4 \;=\; (2\pi\alpha')^2 B_{nm}\partial_i\hat{A}_j\partial_n\partial_j\hat{A}_k\partial_m\partial_k\hat{A}_i$$

$$+ \text{ terms with } \partial^2 A + \text{total derivative} + O(\alpha'^4). \tag{4.21}$$

The first term on the right-hand side of (4.21) provides a non-vanishing contribution to the S-matrix of order $O(B, \zeta^3, k^5)$.

On the commutative side, only terms of the form $O(\partial^2 F^4)$ can produce the same form of the contribution after replacing F with $B + F$. Any term of order $O(\partial^2 F^4)$ can be transformed to the following form using the integration by parts and the Bianchi identity [19]:

$$\mathcal{L} = \sum_{i=1}^{7} b_i J_i, \tag{4.22}$$

where

$$\begin{aligned}
&J_1 = \partial_n F_{ij} \partial_n F_{ji} F_{kl} F_{lk}, \quad J_2 = \partial_n F_{ij} \partial_n F_{jk} F_{kl} F_{li}, \\
&J_3 = F_{ni} F_{im} \partial_n F_{kl} \partial_m F_{lk}, \quad J_4 = \partial_n F_{ni} \partial_m F_{im} F_{kl} F_{lk}, \\
&J_5 = -\partial_n F_{ni} \partial_m F_{ij} F_{jk} F_{km}, \quad J_6 = \partial^2 F_{ij} F_{ji} F_{kl} F_{lk}, \\
&J_7 = \partial^2 F_{ij} F_{jk} F_{kl} F_{li}, \quad \partial^2 F_{ij} = \partial_i \partial_k F_{kj} - \partial_j \partial_k F_{ki}.
\end{aligned} \tag{4.23}$$

The terms J_4, J_5, J_6 and J_7 contain the part $\partial_j F_{ji}$ so that they do not contribute to the S-matrix. This holds after replacing F with $B + F$ since the part $\partial_j F_{ji}$ remains intact in the replacement. Thus we do not need to consider these terms in the search for the term which reproduces the contribution from the term $\hat{T}_4$. On the other hand, the first three coefficients b_1, b_2 and b_3 in this basis do not change under field redefinition and unambiguous [20]. Therefore our goal is to answer the question whether these coefficients are constrained by our assumptions (2.15).

Let us denote the $O(B^n)$ part of J_i with F replaced by $B+F$ as $J_i(B^n)$ following [17]. Explicit expressions of $J_i(B)$ and $J_i(B^2)$ for $i = 1, 2, 3$ are given by

$$\begin{aligned}
&J_1(B) = 2\partial_n F_{ij} \partial_n F_{ji} B_{kl} F_{lk}, \quad J_1(B^2) = \partial_n F_{ij} \partial_n F_{ji} B_{kl} B_{lk}, \\
&J_2(B) = 2B_{ij} F_{jk} \partial_n F_{kl} \partial_n F_{li}, \quad J_2(B^2) = \partial_n F_{ij} \partial_n F_{jk} B_{kl} B_{li}, \\
&J_3(B) = 2B_{ni} F_{im} \partial_n F_{kl} \partial_m F_{lk}, \quad J_3(B^2) = B_{ni} B_{im} \partial_n F_{kl} \partial_m F_{lk}.
\end{aligned} \tag{4.24}$$

It is easily seen that the values of $J_1(B^2)$, $J_2(B^2)$ and $J_3(B^2)$ vanish if they are evaluated at on-shell configurations (3.17) satisfying (3.18). We can also show that the terms $J_1(B)$

and $J_2(B)$ do not contribute to the S-matrix using the formula (3.22). Therefore the term $J_3(B)$ is the only one which contributes to the S-matrix of order $O(B, \zeta^3, k^5)$ on the commutative side, which can be rewritten using (3.22) as follows:

$$\begin{aligned} J_3(B) &= 4B_{ni}\partial_i A_m \partial_n \partial_k A_l \partial_m \partial_l A_k + \text{ terms with } \partial^2 A + \text{total derivative} \\ &= -4B_{ni}\partial_i \partial_l A_m \partial_n \partial_k A_l \partial_m A_k \\ &\quad + \text{a term with } \partial_l A_l + \text{ terms with } \partial^2 A + \text{total derivative} \\ &= -4B_{nm}\partial_i A_j \partial_n \partial_j A_k \partial_m \partial_k A_i \\ &\quad + \text{a term with } \partial \cdot A + \text{ terms with } \partial^2 A + \text{total derivative.} \end{aligned} \tag{4.25}$$

The non-vanishing contribution to the S-matrix from $J_3(B)$ takes the same form as that of $\hat{T}_4$ (4.21) so that it is possible to reproduce the S-matrix from $\hat{T}_4$ by $J_3(B)$ with the following normalization factor:

$$\hat{T}_4 \sim -\frac{1}{4}(2\pi\alpha')^2 J_3(B). \tag{4.26}$$

In addition to J_3, we can add the terms J_1 and J_2 to the effective Lagrangian without violating the assumptions (2.15) since $J_1(B)$ and $J_2(B)$ do not contribute to the S-matrix at the order we are discussing. In general, if a term $f(F)$ satisfies the condition that

$$f(B+F) = f(F) + \text{total derivative using the equation of motion,} \tag{4.27}$$

we can add the term to the effective Lagrangian without violating the assumptions (2.15) at the same order of α' as $f(F)$. We will call (4.27) the *on-shell initial term condition.* Following this terminology, we can say that the terms J_1 and J_2 do not satisfy the initial term condition (3.14) but satisfy the on-shell initial term condition (4.27).

To summarize, the coefficients in front of J_1 and J_2 are not constrained by the assumptions (2.15) since J_1 and J_2 satisfy the on-shell initial term condition. The coefficient in front of J_3 is correlated with that in front of $\hat{T}_4$ on the non-commutative side following the relation (4.26). However, the coefficient in front of $\hat{T}_4$ was arbitrary as we discussed in the preceding subsection so that the coefficient in front of J_3 is also arbitrary. Thus our conclusion is that two-derivative corrections of the form $O(\partial^2 F^4)$ are not constrained at all by the assumptions (2.15) at this order.

This result may seem discouraging in view of our motivation to obtain constraints on the effective Lagrangian. However we do not expect that it holds at higher-order terms in the α' expansion because of the following argument. In general it would become more difficult to satisfy the on-shell initial term condition when the number of field strengths minus the number of derivatives increases in the term under consideration. If we note that the existence of the solutions to the on-shell initial term conditions was essential to our conclusion that there is no constraint on the $O(\partial^2 F^4)$ terms, we can reasonably expect severe constraints on such higher-order terms. We admit, however, that the approach presented in this paper will not be practical in deriving the constraints on the higher-order terms and we need more efficient methods. As an example of promising methods we can refer to the one discussed in [21]. We will get back to this point after discussing the issue on field redefinitions.

There is another comment on our result regarding the relation between the coefficients in front of $\hat{T}_4$ and J_3 (4.26). This provides no information on the effective Lagrangian for the rank-one case since $\hat{T}_4$ vanishes in the commutative limit. However if we succeed in extending our consideration to higher-rank cases, it might be possible to obtain a prediction on a relation between the coefficient in front of the F^3 term and coefficients in $O(D^2 F^4)$ terms.

We should now clarify the relation between the result in this paper and that in [17]. The most general form of $O(\partial^2 F^4)$ terms was derived in [17] from the requirement that $\mathcal{L}(B+F)$ and $\hat{\mathcal{L}}(\hat{F})$ coincide up to total derivative under the assumption that the field redefinition is given by (2.17). The result was that the terms J_1, J_2 and J_3 must appear in the combination that

$$-\frac{1}{4}J_1 + 2J_2 + J_3. \tag{4.28}$$

This was inconsistent with the $O(\partial^2 F^4)$ terms in bosonic string theory derived from the string four-point amplitude [19] or from the two-loop beta function in the open string sigma model [20] which are proportional to**

$$-\frac{1}{4}J_1 - 2J_2 + J_3. \tag{4.29}$$

** This expression is slightly different from (4) in [20] but one of the authors was informed of a misprint in (4) of [20]: the last coefficient b_3 should have sign +.

The conclusion in this paper that no constraint is imposed on $O(\partial^2 F^4)$ terms is trivially consistent with (4.29) and the difference between this conclusion and that in [17] implies that the relation between the two gauge fields $\hat{A}_i$ and A_i in (2.15) does not in general take the form of (2.17) assumed in [17]. In particular, the discrepancy between (4.28) and (4.29) shows that it is indeed the case for bosonic string theory. We will construct a field redefinition which is relevant to bosonic string theory in the next subsection.

4.3 Corrections to the field redefinition

We presented the on-shell initial term condition (4.27) as a necessary condition for a term to be added to the effective Lagrangian without violating the assumptions (2.15) in the preceding subsection. The relation between $\hat{A}_i$ and A_i must be in general modified if we add a term which satisfies the on-shell initial term condition (4.27) but does not satisfy the initial term condition (3.14). As we have seen, the terms J_1 and J_2 are examples of such terms since $J_1(B)$, $J_1(B^2)$, $J_2(B)$ and $J_2(B^2)$ are not total derivative although values of them vanish when evaluated at configurations satisfying the on-shell conditions (3.18). The terms J_4, J_5, J_6 and J_7 also satisfy the on-shell initial term condition, however, they are less interesting than J_1 and J_2 since they do not contribute to the S-matrix. An explicit form of the required field redefinition which relates $\hat{A}_i$ to A_i when we add a term which satisfies the on-shell initial term condition to the effective Lagrangian can be derived in the same way as we did in Section 3.2 but we will not do that for completely general cases. It would be sufficient to demonstrate it for some examples including the one which is relevant to bosonic string theory since the generalization is straightforward.

Let us first consider a case where only J_2 exists in the $O(\alpha'^3)$ part. In particular, the absence of J_3 means that $\hat{T}_4$ is not allowed to exist in $\hat{\mathcal{L}}(\hat{F})$ because of the relation (4.26). Thus there are no $O(\alpha')$ terms in $\hat{\mathcal{L}}(\hat{F})$ under our convention that $\hat{T}_1$ should be redefined away. This simplifies the discussion since the $O(\alpha'^2)$ part in the field redefinition (3.41), which is necessary to satisfy the assumptions (2.15) as we have seen in the preceding section, does not affect $O(\alpha'^3)$ terms under consideration if there are no $O(\alpha')$ terms in $\hat{\mathcal{L}}(\hat{F})$. Furthermore, the $O(\alpha'^2)$ part of $\hat{\mathcal{L}}(\hat{F})$ cannot generate B-dependent terms of order α'^3 which is manifest under our convention (4.18). Therefore the terms $J_2(B)$ and $J_2(B^2)$, which are necessary to realize the B-dependence of the form $B+F$ when we add J_2, must

be generated from the $\hat{F}^2$ term by the $O(\alpha'^3)$ part of the field redefinition of $\hat{A}_i$. Its explicit form is easily derived if we rewrite $J_2(B)$ and $J_2(B^2)$ as follows:

$$\begin{aligned} J_2(B) &= -2\partial_n F_{ni}\partial_j (FBF)_{ji} + \text{total derivative}, && (4.30) \\ J_2(B^2) &= -\partial_n F_{ni}\partial_j (B^2F + FB^2)_{ji} + \text{total derivative}. && (4.31) \end{aligned}$$

It follows from a similar argument to the one used to determine the form (3.41) that the field redefinition

$$\begin{aligned} \hat{A}_i &= A_i + \frac{1}{2}(2\pi\alpha')^2 B_{kl}A_k(\partial_l A_i + F_{li}) \\ &\quad -\frac{1}{4}c_2(2\pi\alpha')^3\partial_j(2FBF + B^2F + FB^2)_{ji} + O(\alpha'^4) \end{aligned} \tag{4.32}$$

generates $c_2(2\pi\alpha')^3(J_2(B) + J_2(B^2))$ from the $\hat{F}^2$ term. To summarize, the two Lagrangians,

$$\begin{aligned} \mathcal{L}(B+F) &= \frac{\sqrt{\det g}}{g_s}\Bigg[\mathrm{Tr}(B+F)^2 + (2\pi\alpha')^2\left[\frac{1}{2}\mathrm{Tr}(B+F)^4 - \frac{1}{8}[\mathrm{Tr}(B+F)^2]^2\right] \\ &\quad + c_2(2\pi\alpha')^3\partial_n(B+F)_{ij}\partial_n(B+F)_{jk}(B+F)_{kl}(B+F)_{li} \\ &\quad + O(\alpha'^4)\Bigg], \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \hat{\mathcal{L}}(\hat{F}) &= \frac{\sqrt{\det G}}{G_s}\Bigg[\mathrm{Tr}(G^{-1}\hat{F} * G^{-1}\hat{F}) \\ &\quad + (2\pi\alpha')^2\left[\frac{1}{2}\mathrm{Tr}(G^{-1}\hat{F})^4_{\text{arbitrary}} - \frac{1}{8}(\mathrm{Tr}(G^{-1}\hat{F})^2)^2_{\text{arbitrary}}\right] \\ &\quad + c_2(2\pi\alpha')^3(\hat{D}_n\hat{F}_{ij} * \hat{D}_n\hat{F}_{jk} * \hat{F}_{kl} * \hat{F}_{li})_{G,\ \text{arbitrary}} + O(\alpha'^4)\Bigg], \end{aligned} \tag{4.34}$$

with an arbitrary ordering of $\hat{D}\hat{F}$'s and $\hat{F}$'s in the $O(\alpha'^3)$ term contracted using G_{ij} as indicated by the subscript, are related by the field redefinition (4.32).

This example shows that α' corrections of $O(B)$ to the field redefinition (3.41) are in general possible. Since

$$B_{ij} = -\frac{1}{(2\pi\alpha')^2}(g\theta g)_{ij} + O(\theta^2), \tag{4.35}$$

this does not take the form of (2.17). Therefore it would be helpful to confirm that (4.32) preserves the gauge equivalence relation (2.16). Let us decompose the field redefinition

(4.32) as follows:

$$\hat{A}_i \to \tilde{A}_i \to A_i, \tag{4.36}$$

where

$$\begin{aligned} \hat{A}_i &= \tilde{A}_i + \frac{1}{2}(2\pi\alpha')^2 B_{kl}\tilde{A}_k(\partial_l \tilde{A}_i + \tilde{F}_{li}) + O(\alpha'^4), && (4.37) \\ \tilde{A}_i &= A_i - \frac{1}{4}c_2(2\pi\alpha')^3 \partial_j(2FBF + B^2F + FB^2)_{ji} + O(\alpha'^4). && (4.38) \end{aligned}$$

By the first part (4.37), the non-commutative gauge field $\hat{A}_i$ is mapped to an ordinary gauge field $\tilde{A}_i$ which respects the ordinary gauge invariance while $\tilde{A}_i$ is mapped to another ordinary gauge field A_i by the second part (4.38) since the difference between $\tilde{A}_i$ and A_i is gauge invariant although it depends on B. This shows that (4.32) preserves the gauge equivalence relation (2.16). In general, the field redefinition (3.41) maps a non-commutative gauge field to an ordinary gauge field but the B-dependence of the effective Lagrangian in terms of the resulting gauge field, $\tilde{A}_i$ in this example, does not take the form of $B+F$. Therefore further B-dependent redefinition like (4.38) is necessary to map it to the gauge field which satisfies the first assumption of (2.15).

The form of the field redefinition (4.32) does not belong to the class of solutions to the gauge equivalence relation (2.16) found in [18]. However there is no contradiction since it was assumed in [18] that Lorentz indices in a mapping from A_i to $\hat{A}_i$ are contracted among derivatives of the gauge field and $\delta\theta^{ij}$ alone while $(g^{-1})^{ij}$ is used in our case (4.32) although it is implicit under our convention (3.5).[††]

Now the extension to cases where other J_i's except J_3 exist in the effective Lagrangian would be straightforward. However if J_3 exists the story becomes slightly complicated because of the presence of $\hat{T}_4$ in $\hat{\mathcal{L}}(\hat{F})$ which accompanies J_3 following the relation (4.26). We have to consider the effects of the $O(\alpha'^2)$ part in the field redefinition (3.41) when it acts on the $O(\alpha')$ term $\hat{T}_4$. Here it is convenient to utilize the results of [17]. Let us review them briefly.

It was shown in [17] that the two Lagrangians,

$$\hat{\mathcal{L}}_1(\hat{F}) = \frac{\sqrt{\det G}}{G_s}\left[\hat{T}_3 + (2\pi\alpha')^2\left(-\frac{1}{4}\hat{J}_1 + 2\hat{J}_2 + \hat{J}_3\right) + O(\alpha'^4)\right], \tag{4.39}$$

[††] We would like to thank I. Kishimoto for clarifying this point.

and

$$\hat{\mathcal{L}}_2(\hat{F}) = \frac{\sqrt{\det G}}{G_s}\left[\hat{T}_1 + (2\pi\alpha')^2\left(\hat{J}_5 - \frac{1}{8}\hat{J}_6 + \frac{1}{2}\hat{J}_7\right) + O(\alpha'^4)\right], \tag{4.40}$$

satisfy $\hat{\mathcal{L}}(\hat{F}) = \mathcal{L}(B+F)$ up to total derivative under the field redefinition (3.41) with the definition of G_s (3.43). Here $\hat{J}_i$'s are the non-commutative counterparts of J_i's with an arbitrary ordering of the fields. We presented the Lagrangians on the non-commutative side because we can uniquely construct their commutative counterparts while the other direction, $\mathcal{L}(B+F) \to \hat{\mathcal{L}}(\hat{F})$, suffers from the ambiguity in the rank-one case discussed in Section 4.1. A linear combination of the two Lagrangians is expressed in our basis $\{\hat{T}_1, \hat{T}_4\}$ as follows:

$$\begin{aligned} \hat{\mathcal{L}}(\hat{F}) \;&=\; \frac{\sqrt{\det G}}{G_s}\Bigg[a\hat{T}_1 + b\hat{T}_4 + a(2\pi\alpha')^2\left(\hat{J}_5 - \frac{1}{8}\hat{J}_6 + \frac{1}{2}\hat{J}_7\right) \\ &\quad -\frac{1}{4}b(2\pi\alpha')^2\left(-\frac{1}{4}\hat{J}_1 + 2\hat{J}_2 + \hat{J}_3 + 2\hat{J}_5 - \frac{1}{4}\hat{J}_6 + \hat{J}_7\right) + O(\alpha'^4)\Bigg]. \end{aligned} \tag{4.41}$$

It was further argued in [17] that (4.41) is the most general form of two-derivative corrections up to this order in the α' expansion which satisfy $\hat{\mathcal{L}}(\hat{F}) = \mathcal{L}(B+F)$ up to total derivative under the field redefinition (3.41) with the definition of G_s (3.43).[‡‡] To see that it is the case, it is helpful to notice that if there is another Lagrangian

$$\begin{aligned} \hat{\mathcal{L}}'(\hat{F}) \;&=\; \frac{\sqrt{\det G}}{G_s}\Bigg[a\hat{T}_1 + b\hat{T}_4 \\ &\quad + O(\alpha'^2) \text{ terms different from those of } \hat{\mathcal{L}}(\hat{F}) + O(\alpha'^4)\Bigg], \end{aligned} \tag{4.42}$$

which also satisfies $\hat{\mathcal{L}}'(\hat{F}) = \mathcal{L}'(B+F)$ up to total derivative under the field redefinition (3.41) with the definition of G_s (3.43), then the difference $\mathcal{L}'(F) - \mathcal{L}(F)$ must be a solution of the form $O(\partial^2 F^4)$ to the initial term condition (3.14). Thus the question whether (4.41) is the most general form reduces to the one whether there are solutions of the form $O(\partial^2 F^4)$ to the initial term condition (3.14). Regarding the latter question, it was shown [17] that the condition that $O(\partial^2 F^4)$ terms must be proportional to the combination that

$$\mathcal{F}(F) \equiv -\frac{1}{4}J_1 + 2J_2 + J_3 + 2J_5 - \frac{1}{4}J_6 + J_7, \tag{4.43}$$

[‡‡] The argument for proving this statement developed in Section 3 of [17] was incorrect as explained in the note added at the end of hep-th/9909132 v2.

is necessary to satisfy the initial term condition (3.14). It is difficult to see whether $\mathcal{F}(F)$ satisfies the initial term condition by a direct calculation, however, we can obtain the answer by an indirect argument in the following way. From (4.20) and the fact that (4.41) with $a = 0$ satisfies $\hat{\mathcal{L}}(\hat{F}) = \mathcal{L}(B+F)$ up to total derivative under the field redefinition (3.41) with the definition of G_s (3.43), it follows that

$$-\frac{1}{4}\mathcal{F}(B+F) = \frac{1}{2}B_{nm}F_{ij}\partial_n F_{jk}\partial_m F_{ki} - \frac{1}{4}\mathcal{F}(F) + \text{total derivative}, \tag{4.44}$$

which implies that $\mathcal{F}(F)$ does not satisfy the initial term condition (3.14). Now that the only remaining possibility was denied, the statement that there is no solution of the form $O(\partial^2 F^4)$ to the initial term condition (3.14) was shown and this implies that (4.41) is the most general form of two-derivative corrections up to this order in the α' expansion which satisfy $\hat{\mathcal{L}}(\hat{F}) = \mathcal{L}(B+F)$ up to total derivative under the field redefinition (3.41) with the definition of G_s (3.43).

This result provides us with a good starting point for the case where J_3 is non-vanishing. Namely, the Lagrangian

$$\begin{aligned}\hat{\mathcal{L}}(\hat{F}) \;=\; \frac{\sqrt{\det G}}{G_s}\Big[&\mathrm{Tr}(G^{-1}\hat{F} * G^{-1}\hat{F}) + b(2\pi\alpha')\hat{T}_4 \\ &+(2\pi\alpha')^2\Big[\frac{1}{2}\mathrm{Tr}(G^{-1}\hat{F})^4_{\text{arbitrary}} - \frac{1}{8}(\mathrm{Tr}(G^{-1}\hat{F})^2)^2_{\text{arbitrary}}\Big] \\ &-\frac{1}{4}b(2\pi\alpha')^3\left(-\frac{1}{4}\hat{J}_1 + 2\hat{J}_2 + \hat{J}_3 + 2\hat{J}_5 - \frac{1}{4}\hat{J}_6 + \hat{J}_7\right) \\ &+ O(\alpha'^4)\Big],\end{aligned} \tag{4.45}$$

and $\mathcal{L}(B+F)$ constructed from $\hat{\mathcal{L}}(\hat{F})$ are related by the field redefinition (3.41). If we want to change the coefficients in font of J_i's except J_3, we should modify the form of the field redefinition at order α'^3 appropriately as in the preceding example where only J_2 exists.

As an interesting example of such cases, let us derive the form of the field redefinition which is relevant to bosonic string theory. As we mentioned in the preceding subsection, the coefficients in front of J_1, J_2 and J_3 calculated in bosonic string theory are proportional to (4.29) [19, 20]. This corresponds to adding $b(2\pi\alpha')^3\hat{J}_2$ to (4.45) so that the form of the

field redefinition is modified to

$$\begin{aligned}\hat{A}_i &= A_i + \frac{1}{2}(2\pi\alpha')^2 B_{kl} A_k (\partial_l A_i + F_{li}) \\ &\quad -\frac{1}{4} b (2\pi\alpha')^3 \partial_j (2FBF + B^2 F + FB^2)_{ji} + O(\alpha'^4).\end{aligned} \tag{4.46}$$

If we further change the coefficients in front of J_4, J_5, J_6 and J_7 which do not affect the S-matrix, the form of the field redefinition (4.46) itself is modified correspondingly. However we cannot make the $O(\alpha'^3)$ terms vanish since (4.29) does not take the general form (4.41) in the absence of the $O(\alpha'^3)$ terms. Thus the corrections to the field redefinition (3.41) are not only possible in principle but also realized actually in bosonic string theory. For superstring theory, it was found that the coefficients in front of J_1, J_2 and J_3 vanish [19] so that corrections to the field redefinition (3.41) at order α'^3 are not required. However there is no general argument that it persists to higher orders in the α' expansion. We should keep such possibility of corrections in mind when we use properties of the field redefinition which relates the non-commutative gauge field to the ordinary one. In particular, it would be important to note that corrections of $O(B) \sim O(\theta)$ modify the differential equation of $\delta\hat{A}(\theta)$ introduced in [12] for more general descriptions of the system in terms of non-commutative gauge theory.

5. Conclusions and discussions

We considered the constraints on the effective Lagrangian of the gauge field on a single D-brane in flat space-time imposed by the compatibility of the description by non-commutative gauge theory $\hat{\mathcal{L}}(\hat{F})$ with that by ordinary gauge theory $\mathcal{L}(B+F)$ in the presence of a constant B field background. We presented a systematic method under the α' expansion to derive the constraints based on the assumptions (2.15) alone without assuming the form of the field redefinition which relates the non-commutative gauge field $\hat{A}_i$ to the ordinary one A_i.

By applying this method to two-derivative corrections to the DBI Lagrangian, we established the equivalence of the two descriptions for a larger class of Lagrangians. In particular it contains the effective Lagrangian of bosonic string theory and thus the puzzle in bosonic strings found in the previous works was resolved and the equivalence in this

case was first made consistent.

In resolving the puzzle it was essential to observe that the gauge-invariant but B-dependent corrections to the field redefinition (2.17) are in general necessary for the compatibility. They were not considered previously because it was assumed that the metric g_{ij} does not appear in the field redefinition which relates $\hat{A}_i$ to A_i, however we showed that they must exist for the case of bosonic string theory. It should be emphasized that it was crucial to reach this observation that we did not assume the form of the field redefinition when we derive the constraints.

It is sometimes said that the form of the field redefinition which relates $\hat{A}_i$ to A_i can be determined by solving the differential equation derived from the gauge equivalence relation in [12] up to the ambiguities found in [18]. However our result clearly shows that it is no longer the case if we allow the metric g_{ij} to appear in the field redefinition as in the case of bosonic string theory. Even the form of the differential equation itself can be modified and the form of the field redefinition is hardly constrained by the gauge equivalence relation without the assumption. In the superstring case the field redefinition of the form (2.17) can be consistent up to two derivatives [21] and may not be corrected. However, as far as we are aware of, there is no argument which justifies the assumption in the superstring case that the metric g_{ij} does not appear in the field redefinition which relates $\hat{A}_i$ to A_i.

We believe that we have elucidated the mechanism to constrain the effective Lagrangian of the gauge fields on D-branes using non-commutative gauge theory. Since we presented a systematic method to obtain the constraints in the α' expansion, it is in principle possible to calculate the general form of the effective Lagrangian which satisfies the assumptions (2.15) up to an arbitrary order in α'. Furthermore our study implies that the number of the free parameters in the general form is equal to or less than the number of solutions to the initial term condition (3.14) plus that of nontrivial solutions to the on-shell initial term condition (4.27). Therefore if there are no solutions to these conditions in the two-derivative terms at higher order in α', for example, this implies that the form of the two-derivative terms is in principle determined uniquely by the requirement of the compatibility up to the parameters in front of J_1, J_2 and J_3. However we admit that the approach adopted in the present paper is not practically useful to proceed to the higher

orders to obtain the constraints or the explicit form of the terms as we mentioned in Section 4.2. Regarding this issue, a general method to construct $2n$-derivative terms to all orders in α' which satisfy the compatibility of the two descriptions in the approximation of neglecting $(2n+2)$-derivative terms when the field redefinition takes the form of (2.17) was presented in [21]. It is therefore necessary to extend the method to apply it to more general cases where there are corrections to the field redefinition of the form (2.17) such as the case of bosonic string theory. If we could succeed in such generalization, it would be expected that it will provide us with a new powerful method to study the dynamics of D-branes. For developments in this direction, the simplified Seiberg-Witten map considered in [22] and [23] may be useful because of its geometric nature although we should clarify its meaning for our approach. See also a related work [24]. How to construct actions which are invariant under the simplified map was recently discussed in [25]. In addition, it is interesting to combine our approach with consideration of supersymmetry and string dualities. It will probably provide us with further constraints.

We only considered the constraints on the effective Lagrangian at the lowest order in the expansion with respect to the string coupling constant g_s in this paper. There seems to be no crucial obstruction to the extension of our approach to higher orders in g_s although some modifications may be required. An issue related to this kind of extension was discussed in [26]. Furthermore, although the assumptions (2.15) were derived from the action of the sigma model, they are not related to the expansions with respect to α' and g_s once extracted. It would be interesting if we could obtain some non-perturbative information on the dynamics of D-branes from them. Of course it might be the case that there are limitations of the description in terms of non-commutative gauge theory at non-perturbative level and it is important to investigate them.

Another important extension of our approach is to consider higher-rank gauge theory. It would be interesting if we could obtain some insight into the non-Abelian generalization of the DBI Lagrangian [27]. Although we foresee possible complication originated in its non-Abelian nature which exists even on the side of ordinary gauge theory, it will be worth investigating in view of the various important developments which have been made by the super Yang-Mills theory in the description of multi-body systems of D-branes.

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Appendix A. Determination of the $O(\alpha'^0)$ part of G_s

In this appendix, we determine the $O(\alpha'^0)$ part of G_s, namely the constant t in (3.11), by the assumptions (2.15). Since the normalizations of A_i and $\hat{A}_i$ do not coincide when $t \neq 1$ as can be seen from (3.12), we should be careful when evaluating the S-matrix. A safer approach is to make calculations after rescaling both fields such that their normalizations coincide. We denote the normalized fields by $\mathbf{A}_i$ and $\hat{\mathbf{A}}_i$:

$$\mathbf{A}_i = \frac{A_i}{\sqrt{g_s}}, \quad \hat{\mathbf{A}}_i = \frac{\hat{A}_i}{\sqrt{G_s}}. \tag{A.1}$$

If we define the field strengths of the normalized fields as

$$\mathbf{F}_{ij} \equiv \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i, \tag{A.2}$$

$$\hat{\mathbf{F}}_{ij} \equiv \partial_i \hat{\mathbf{A}}_j - \partial_j \hat{\mathbf{A}}_i - i\sqrt{G_s}\hat{\mathbf{A}}_i * \hat{\mathbf{A}}_j + i\sqrt{G_s}\hat{\mathbf{A}}_j * \hat{\mathbf{A}}_i, \tag{A.3}$$

the effective Lagrangians $\mathcal{L}(B+F)$ (3.7) and $\hat{\mathcal{L}}(\hat{F})$ (3.8) can be rewritten as follows:

$$\begin{aligned}
\mathcal{L}(B+F) &= \sqrt{\det g}\,\mathrm{Tr}\left(\frac{B}{\sqrt{g_s}} + \mathbf{F}\right)^2 + O(\alpha') \\
&= \sqrt{\det g}\,\mathrm{Tr}\,\mathbf{F}^2 + \text{total derivative} + O(\alpha'),
\end{aligned} \tag{A.4}$$

$$\hat{\mathcal{L}}(\hat{F}) = \sqrt{\det G}\,\mathrm{Tr}(G^{-1}\hat{\mathbf{F}} * G^{-1}\hat{\mathbf{F}}) + O(\alpha'). \tag{A.5}$$

It is clear from these expressions that the normalized fields $\mathbf{A}_i$ and $\hat{\mathbf{A}}_i$ coincide at the lowest order in α':

$$\hat{\mathbf{A}}_i = \mathbf{A}_i + O(\alpha'). \tag{A.6}$$

Following the procedure presented in Section 3.1, the F^4 terms can be determined in this case as well. The evaluation of the Lagrangian on the non-commutative side in the α' expansion is given in terms of the normalized field $\hat{\mathbf{A}}_i$ by

$$\begin{aligned}
&\sqrt{\det G}\,\mathrm{Tr}(G^{-1}\hat{\mathbf{F}} * G^{-1}\hat{\mathbf{F}}) \\
= &\sqrt{\det g}\Big[(\partial_i\hat{\mathbf{A}}_j - \partial_j\hat{\mathbf{A}}_i)(\partial_j\hat{\mathbf{A}}_i - \partial_i\hat{\mathbf{A}}_j) - 4(2\pi\alpha')^2\sqrt{G_s}B_{kl}\partial_k\hat{\mathbf{A}}_i\partial_l\hat{\mathbf{A}}_j\partial_j\hat{\mathbf{A}}_i \\
&+2(2\pi\alpha')^2(B^2)_{ij}(\partial_j\hat{\mathbf{A}}_k - \partial_k\hat{\mathbf{A}}_j)(\partial_k\hat{\mathbf{A}}_i - \partial_i\hat{\mathbf{A}}_k) \\
&-\frac{1}{2}(2\pi\alpha')^2\mathrm{Tr}B^2(\partial_i\hat{\mathbf{A}}_j - \partial_j\hat{\mathbf{A}}_i)(\partial_j\hat{\mathbf{A}}_i - \partial_i\hat{\mathbf{A}}_j) + O(\alpha'^4)\Big].
\end{aligned} \tag{A.7}$$

The relevant terms on the commutative side are

$$
\begin{aligned}
& \frac{\sqrt{\det g}}{g_s}\mathrm{Tr}(B+F)^4 = \sqrt{\det g}\, g_s \mathrm{Tr}\left(\frac{B}{\sqrt{g_s}}+\mathbf{F}\right)^4 \\
= \quad & \sqrt{\det g}\,\left[g_s \mathrm{Tr}\,\mathbf{F}^4 + 4\sqrt{g_s}\,\mathrm{Tr}B\mathbf{F}^3 + O(B^2)\right], \qquad \text{(A.8)} \\
& \frac{\sqrt{\det g}}{g_s}[\mathrm{Tr}(B+F)^2]^2 = \sqrt{\det g}\, g_s \left[\mathrm{Tr}\left(\frac{B}{\sqrt{g_s}}+\mathbf{F}\right)^2\right]^2 \\
= \quad & \sqrt{\det g}\,\left[g_s(\mathrm{Tr}\,\mathbf{F}^2)^2 + 4\sqrt{g_s}\,\mathrm{Tr}B\mathbf{F}\,\mathrm{Tr}\,\mathbf{F}^2 + O(B^2)\right]. \qquad \text{(A.9)}
\end{aligned}
$$

The requirement that both Lagrangians $\hat{\mathcal{L}}(\hat{F})$ and $\mathcal{L}(B+F)$ should produce the same S-matrix of order $O(B,\zeta^3,k^3)$ determines the form of the Lagrangian $\mathcal{L}(B+F)$ in a completely parallel way to Section 3.1. The result is as follows:

$$
\begin{aligned}
\mathcal{L}(B+F) \quad = \quad & \sqrt{\det g}\left[\mathrm{Tr}\left(\frac{B}{\sqrt{g_s}}+\mathbf{F}\right)^2 + (2\pi\alpha')^2\sqrt{G_s\, g_s}\left[\frac{1}{2}\mathrm{Tr}\left(\frac{B}{\sqrt{g_s}}+\mathbf{F}\right)^4\right.\right. \\
& \left.\left. -\frac{1}{8}\left(\mathrm{Tr}\left(\frac{B}{\sqrt{g_s}}+\mathbf{F}\right)^2\right)^2\right] + O(\alpha'^4) + \text{derivative corrections}\right] \\
= \quad & \frac{\sqrt{\det g}}{g_s}\left[\mathrm{Tr}(B+F)^2 + \sqrt{t}\,(2\pi\alpha')^2\left[\frac{1}{2}\mathrm{Tr}(B+F)^4\right.\right. \\
& \left.\left. -\frac{1}{8}[\mathrm{Tr}(B+F)^2]^2\right] + \; O(\alpha'^4) + \text{derivative corrections}\right], \qquad \text{(A.10)}
\end{aligned}
$$

where we used t defined by (3.11). The $O(B)$ part of this Lagrangian coincides with that of (A.7) up to field redefinition since both produce the same S-matrix, but it may not the case for the $O(B^2)$ part. The $O(\alpha'^2)$ part of the Lagrangian (A.10) expanded with respect to B is given by

$$
\begin{aligned}
& \sqrt{\det g}\,(2\pi\alpha')^2\sqrt{G_s\, g_s}\left[\frac{1}{2}\mathrm{Tr}\left(\frac{B}{\sqrt{g_s}}+\mathbf{F}\right)^4 - \frac{1}{8}\left(\mathrm{Tr}\left(\frac{B}{\sqrt{g_s}}+\mathbf{F}\right)^2\right)^2\right] \\
= \quad & \sqrt{\det g}\,(2\pi\alpha')^2\left[\sqrt{t}\, g_s\left[\frac{1}{2}\mathrm{Tr}\,\mathbf{F}^4 - \frac{1}{8}(\mathrm{Tr}\,\mathbf{F}^2)^2\right]\right. \\
& +\sqrt{t\, g_s}\left(2\mathrm{Tr}B\mathbf{F}^3 - \frac{1}{2}\mathrm{Tr}B\mathbf{F}\,\mathrm{Tr}\,\mathbf{F}^2\right) \\
& \left. +\sqrt{t}\left(2\mathrm{Tr}B^2\mathbf{F}^2 - \frac{1}{4}\mathrm{Tr}B^2\,\mathrm{Tr}\,\mathbf{F}^2\right) + \text{total derivative} + \text{const.}\right]. \qquad \text{(A.11)}
\end{aligned}
$$

The $O(B^2)$ part does not coincide with that of (A.7). The difference in the $\mathrm{Tr}B^2\mathrm{Tr}F^2$ term can be absorbed by field redefinition as we explained in Section 3.3 so that it is irrelevant to the determination of t, whereas the values of the $\mathrm{Tr}B^2F^2$ term evaluated at on-shell configurations satisfying (3.18) do not vanish so that if there is a difference in the term it cannot be redefined away. Therefore the $\mathrm{Tr}B^2F^2$ terms in (A.7) and (A.11) must coincide, which determines the value of t. The result is

$$t = 1. \tag{A.12}$$

Appendix B. Determination of the $O(\alpha'^2)$ part of G_s

As we discussed in the last part of Section 3, the consideration at order α'^2 is not sufficient to determine the $O(\alpha'^2)$ part of G_s and that of the field redefinition which relates $\hat{A}_i$ to A_i uniquely but allows the following ambiguity:

$$G_s = g_s\left[1 + \frac{c-1}{4}(2\pi\alpha')^2\mathrm{Tr}B^2 + O(\alpha'^4)\right], \tag{B.1}$$

$$\hat{A}_i = A_i + \frac{1}{2}(2\pi\alpha')^2 B_{kl}A_k(\partial_l A_i + F_{li}) + \frac{c}{8}(2\pi\alpha')^2\mathrm{Tr}B^2 A_i + O(\alpha'^4), \tag{B.2}$$

where c is an undetermined constant. In this appendix, we determine the value of c by the consideration at order α'^4.

We should first note that whatever ordering of the field strengths we choose, the $*$ product between the field strengths in the $\hat{F}^4$ terms of (3.29) does not affect $O(\alpha'^4)$ terms, namely,

$$(2\pi\alpha')^2\mathrm{Tr}(G^{-1}\hat{F})^4_{\mathrm{arbitrary}} = (2\pi\alpha')^2\mathrm{Tr}(G^{-1}\hat{F}G^{-1}\hat{F}G^{-1}\hat{F}G^{-1}\hat{F}) + O(\alpha'^6), \tag{B.3}$$

$$(2\pi\alpha')^2(\mathrm{Tr}(G^{-1}\hat{F})^2)^2_{\mathrm{arbitrary}} = (2\pi\alpha')^2[\mathrm{Tr}(G^{-1}\hat{F}G^{-1}\hat{F})]^2 + O(\alpha'^6), \tag{B.4}$$

where the product between the field strengths on the right-hand sides of these expressions is the ordinary one, not the $*$ product. Now the $\hat{F}^4$ terms in (3.29) are evaluated in the α' expansion using (3.1), (3.3), (B.1) and (B.2) as follows:

$$\begin{aligned}
&\frac{\sqrt{\det G}}{G_s}\left[\frac{1}{2}(2\pi\alpha')^2\mathrm{Tr}(G^{-1}\hat{F})^4_{\mathrm{arbitrary}} - \frac{1}{8}(2\pi\alpha')^2(\mathrm{Tr}(G^{-1}\hat{F})^2)^2_{\mathrm{arbitrary}}\right] \\
= \ &\frac{\sqrt{\det g}}{g_s}\Bigg[\frac{1}{2}(2\pi\alpha')^2\mathrm{Tr}F^4 - \frac{1}{8}(2\pi\alpha')^2(\mathrm{Tr}F^2)^2
\end{aligned}$$

$$
\begin{aligned}
&+(2\pi\alpha')^4\Big[2\mathrm{Tr}BF^5-\frac{1}{4}\mathrm{Tr}BF\mathrm{Tr}F^4-\frac{1}{2}\mathrm{Tr}BF^3\mathrm{Tr}F^2+\frac{1}{16}\mathrm{Tr}BF(\mathrm{Tr}F^2)^2\Big]\\
&+(2\pi\alpha')^4\Big[2\mathrm{Tr}B^2F^4+\frac{c-1}{8}\mathrm{Tr}B^2\mathrm{Tr}F^4-\frac{1}{2}\mathrm{Tr}B^2F^2\mathrm{Tr}F^2+\frac{1-c}{32}\mathrm{Tr}B^2(\mathrm{Tr}F^2)^2\Big]\\
&+\text{ total derivative}+O(\alpha'^6)\Bigg]. \qquad \text{(B.5)}
\end{aligned}
$$

Since no other terms can produce $O(BF^5)$ terms, (B.5) gives the complete form of them on the non-commutative side. On the other hand, there are three sources for $O(BF^5)$ terms on the commutative side, which are

$$
\begin{aligned}
\mathrm{Tr}(B+F)^6 &= \mathrm{Tr}F^6+6\mathrm{Tr}BF^5+O(B^2), \qquad \text{(B.6)}\\
\mathrm{Tr}(B+F)^2\mathrm{Tr}(B+F)^4 &= \mathrm{Tr}F^2\mathrm{Tr}F^4\\
&\quad +2\mathrm{Tr}BF\mathrm{Tr}F^4+4\mathrm{Tr}F^2\mathrm{Tr}BF^3+O(B^2), \qquad \text{(B.7)}\\
\Big[\mathrm{Tr}(B+F)^2\Big]^3 &= (\mathrm{Tr}F^2)^3+6\mathrm{Tr}BF(\mathrm{Tr}F^2)^2+O(B^2). \qquad \text{(B.8)}
\end{aligned}
$$

By comparison, we can see that the $O(BF^5)$ terms in (B.5) are reproduced by the following terms in $\mathcal{L}(B+F)$:

$$
\frac{(2\pi\alpha')^4\sqrt{\det g}}{g_s}\left[\frac{1}{3}\mathrm{Tr}(B+F)^6-\frac{1}{8}\mathrm{Tr}(B+F)^2\mathrm{Tr}(B+F)^4+\frac{1}{96}\Big[\mathrm{Tr}(B+F)^2\Big]^3\right]. \qquad \text{(B.9)}
$$

These are precisely the terms needed to take the form of the DBI Lagrangian (3.27) under our normalization convention. To show that this is the unique structure of the F^6 terms consistent with the assumptions (2.15), we must verify that no solution to the on-shell initial term condition (4.27) is possible in the F^6 terms. However, even if there exist such solutions, although we believe that there is none, the resulting ambiguity does not affect the determination of the $O(\alpha'^2)$ part of G_s since solutions to the on-shell initial term condition by definition do not contribute to the B-dependent part of the S-matrix. Thus the argument which has been made so far is sufficient for the determination.

Now let us compare the $O(B^2)$ part of (B.9),

$$
\begin{aligned}
\frac{(2\pi\alpha')^4\sqrt{\det g}}{g_s}\Big[&2\mathrm{Tr}B^2F^4-\frac{1}{8}\mathrm{Tr}B^2\mathrm{Tr}F^4-\frac{1}{2}\mathrm{Tr}F^2\mathrm{Tr}B^2F^2+\frac{1}{32}\mathrm{Tr}B^2(\mathrm{Tr}F^2)^2\\
&+2\mathrm{Tr}BFBF^3+\mathrm{Tr}BF^2BF^2-\mathrm{Tr}BF\mathrm{Tr}BF^3\\
&-\frac{1}{4}\mathrm{Tr}F^2\mathrm{Tr}BFBF+\frac{1}{8}\mathrm{Tr}F^2(\mathrm{Tr}BF)^2\Big], \qquad \text{(B.10)}
\end{aligned}
$$

with the corresponding one on the non-commutative side. In addition to the $O(B^2)$ part of (B.5) at order α'^4, there is the following contribution from the $\hat{F}^2$ term:

$$\frac{(2\pi\alpha')^4\sqrt{\det g}}{g_s}\Big[\mathrm{Tr}BF^2BF^2 + 2A_kB_{kl}\partial_lF_{ij}(FBF)_{ji} + A_kB_{kl}\partial_lF_{ij}A_nB_{nm}\partial_mF_{ji} \\ +2B_{kl}B_{nm}A_n(\partial_mA_i + F_{mi})\partial_lA_j\partial_kF_{ji} + O(B^3) + \text{total derivative}\Big]. \quad \text{(B.11)}$$

By the comparison with respect to the terms $\mathrm{Tr}B^2\mathrm{Tr}F^4$ and $\mathrm{Tr}B^2(\mathrm{Tr}F^2)^2$ which obviously contribute to the S-matrix, the value of the constant c is determined. The result is

$$c = 0. \quad \text{(B.12)}$$

To complete the argument of the $O(B^2)$ part at order α'^4, it is necessary to verify that the difference,

$$\frac{(2\pi\alpha')^4\sqrt{\det g}}{g_s}\Big[2\mathrm{Tr}BFBF^3 - \mathrm{Tr}BF\mathrm{Tr}BF^3 - \frac{1}{4}\mathrm{Tr}F^2\mathrm{Tr}BFBF + \frac{1}{8}\mathrm{Tr}F^2(\mathrm{Tr}BF)^2 \\ -2A_kB_{kl}\partial_lF_{ij}(FBF)_{ji} - A_kB_{kl}\partial_lF_{ij}A_nB_{nm}\partial_mF_{ji} \\ -2B_{kl}B_{nm}A_n(\partial_mA_i + F_{mi})\partial_lA_j\partial_kF_{ji}\Big], \quad \text{(B.13)}$$

does not contribute to the S-matrix and is absorbed by a field redefinition of $\hat{A}_i$ at order α'^4. This is an interesting problem itself since it is related to the $O(\theta^2)$ part of (2.17). However it is easily seen that it is irrelevant to the determination of the constant c at any rate because none of the terms in (B.13) have the structure of the contraction $\mathrm{Tr}B^2 = B_{ij}B_{ji}$ which was relevant to the determination.

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